and the sail assembly. Practical aspects of mathematical modeling and simulation have ben emphasized. This Note presents a picture of elastoplastic behavior of the lanyard under different test conditions. A mathematical approach has been developed for studying cumulative deformation and assessing the margin of safety in terms of number of cycles before failure.

Appendix: Ground Simulation Model

The system under consideration is a CLB with no preload on the tip plate, without pop-out spring and sail assembly. This is released under gravity for a jump distance of 0.023 m with the lanyard becoming taut, when the system reaches this jump distance. Then the system is retracted and adjusted such that the slack lanyard will be taut for a jump distance of 0.043 m, when the system is released. The system is retracted and the preceding procedure is repeated for the jump distances of 0.063 m, in steps of 0.01 m, until the lanyard fails. In the test, the lanyard failed for the 0.113-m jump distance with all cumulative damages due to earlier tests.

The mathematical model developed for this configuration is a single DOF system with bilinear boom stiffness characteristics. The deformation of the lanyard is calculated based on the force-deflection characteristics. The cumulative permanent deformation of the lanyard calculated by this model exceeded the allowable deformation for the jump distance of 0.103 m indicating lanyard failure. This is in agreement with test results which showed the failure for a jump distance of 0.113 m.

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Solutions to Parameter Optimal Control

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Introduction

PARAMETER optimal control optimizes parameters subject to differential constraints. It is between regular optimal control and parameter optimization. It has many applications. In particular, trajectory optimization problems can be converted into parameter optimal control with various control parameterizations.

Some work has been done on parameter optimal control. Vincent and Grantham[1] studied necessary conditions using a parameter optimization method. Stengel[2] discussed possible applications of the problem. Rogers[3] developed a first-order numerical method.

This Note uses two approaches and derives the first- and second-order conditions for unconstrained parameter optimal control. The obtained gradient expressions can be used effectively in numerical solutions.

Problem Statement

A parameter optimal control problem seeks to minimize

$$I = \phi(x_f, \pi) + \int_{t_0}^{t_f} L(t, x, \pi)\, dt \tag{1}$$

subject to

$$\dot{x} = f(t, x, \pi) \tag{2}$$

where $x(t)$ is an $n \times 1$ state vector, π is a $p \times 1$ parameter vector, and both ϕ and L are scalars. We assume that t_0 and t_f are fixed, and $x(t_0) = x_0$ is specified. π is constant over $[t_0, t_f]$.

Parameter optimal control problems appear naturally in engineering. Engine parameter optimizations, for example, are subject to motion dynamics. A trajectory optimization problem can be solved efficiently with parameter optimal control. Parameter optimal control can also be used to determine feedback gains in a nonlinear system. In these applications, the gradient expressions derived in this Note require many fewer numerical integrations than a direct digital differentiation.

Solution Methods: Overview

There are two basic approaches to solve a parameter optimal control problem. Parameter optimal control can be viewed as a special optimal control problem that only uses parameters. Also, it can be seen as a special parameter optimization problem with differential constraints. These two interpretations provide two basic solution methods.

In the optimal control approach, the differential constraints are adjoined to the performance index through the use of Lagrange multiplier functions. As a result, one obtains simple expressions for both the first- and second-order gradients. The parameter optimization approach produces consistent results but does not lead to clean expressions for the second-order gradient.

One can combine the two approaches in constructing numerical methods. Nonlinear programming algorithms can be used directly with the simple expressions of the first- and second-order gradients from the optimal control approach.

Necessary Conditions

In the optimal control approach, we adjoin Eq. (2) into the performance index I in Eq. (1). The augmented cost functional is

$$J = \phi(x_f, \pi) + \int_{t_0}^{t_f} \left[H(t, x, \pi, \lambda) - \lambda^T \dot{x}\right] dt \tag{3}$$

where $\lambda(t)$ is an $n \times 1$ Lagrange multiplier function, and H is the Hamiltonian defined as

$$H(t, x, \pi, \lambda) \triangleq L(t, x, \pi) + \lambda^T f(t, x, \pi) \tag{4}$$

The first variation of J is

$$\begin{aligned} \delta J &= \phi_{x_f} \delta x_f + \phi_\pi \delta\pi \\ &\quad + \int_{t_0}^{t_f} \left[H_x\, \delta x + H_\pi \delta\pi - \lambda^T \delta\dot{x} + \delta\lambda^T (f - \dot{x})\right] dt \\ &= (\phi_{x_f} - \lambda_f^T)\delta x_f + \lambda_0^T \delta x_0 \\ &\quad + \left[\phi_\pi + \int_{t_0}^{t_f} H_\pi\, dt\right]\delta\pi + \int_{t_0}^{t_f} (H_x + \dot{\lambda}^T)\delta x\, dt \end{aligned} \tag{5}$$

Noting $\delta x_0 = 0$ and choosing

$$\dot{\lambda} = -H_x^T \qquad \lambda(t_f) = \phi_{x_f}^T \tag{6}$$

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we have

$$\delta J = \left[\phi_\pi + \int_{t_0}^{t_f} H_\pi \, \mathrm{d}t \right] \delta\pi \tag{7}$$

Since $\delta\pi$ is arbitrary, it is necessary that at optimum

$$J_\pi \triangleq \phi_\pi + \int_{t_0}^{t_f} H_\pi \, \mathrm{d}t = 0 \tag{8}$$

In the parameter optimization approach, we write the problem symbolically as

$$I(\pi) = \phi(x_f(\pi), \pi) + \int_{t_0}^{t_f} L(t, x(t, \pi), \pi) \, \mathrm{d}t \tag{9}$$

where $x(t, \pi)$ is determined from Eq. (2).

Small changes in parameters and those in states are related by

$$\delta\dot{x} = f_x \delta x + f_\pi \delta\pi \tag{10}$$

and $\delta x(t_0) = 0$. The linearity of Eq. (10) suggests

$$\delta x(t) = A(t, t_0)\delta\pi \Rightarrow \frac{\partial x}{\partial \pi} = A(t, t_0) \tag{11}$$

where $A(t, t_0) \in R^{n \times p}$ satisfies

$$\dot{A} = f_x A + f_\pi \qquad A(t_0, t_0) = 0 \tag{12}$$

Applying the Leibnitz rule, we obtain

$$I_\pi = \phi_\pi + \phi_{x_f} A_f + \int_{t_0}^{t_f} (L_\pi + L_x A) \, \mathrm{d}t \tag{13}$$

In this expression, $A_f = A(t_f, t_0)$ and $A = A(t, t_0)$. At optimum, $I_\pi = 0$.

The two necessary conditions are consistent. In fact

$$I_\pi = \phi_\pi + \int_{t_0}^{t_f} (L_\pi + \lambda^T f_\pi) \, \mathrm{d}t = J_\pi$$

Sufficient Conditions

One can also derive consistent sufficient conditions using the previous two approaches. The second-order gradient expression from the parameter optimization approach needs the variations of $A(t)$ matrix and is messy. Using the optimal control approach, we have $\delta^2 J = \delta\pi^T P \delta\pi$, where

$$P = \phi_{\pi\pi} + \phi_{\pi x_f} A_f + A_f^T \phi_{x_f \pi} + A_f^T \phi_{x_f x_f} A_f + \int_{t_0}^{t_f} \left[H_{\pi\pi} + H_{\pi x} A + A^T H_{x\pi} + A^T H_{xx} A \right] \mathrm{d}t \tag{14}$$

Therefore, the sufficient conditions for a local minimum include $P > 0$. In particular, it is not sufficient for a parameter optimal control problem to have

$$\phi_{\pi\pi} + \int_{t_0}^{t_f} H_{\pi\pi} \, \mathrm{d}t > 0 \tag{15}$$

In comparison, the sufficient conditions for a regular unconstrained optimal control problem include the strengthened Legendre-Clebsch condition and the no conjugate point condition.[4]

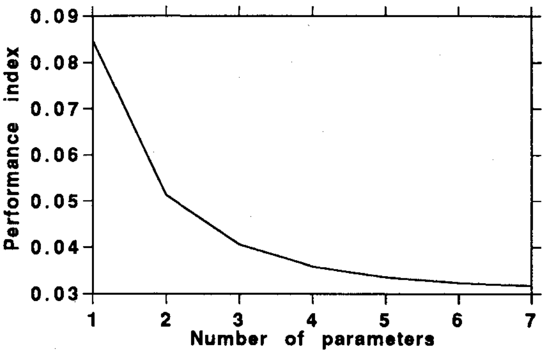

Fig. 1 Performance index as a function of number of parameters.

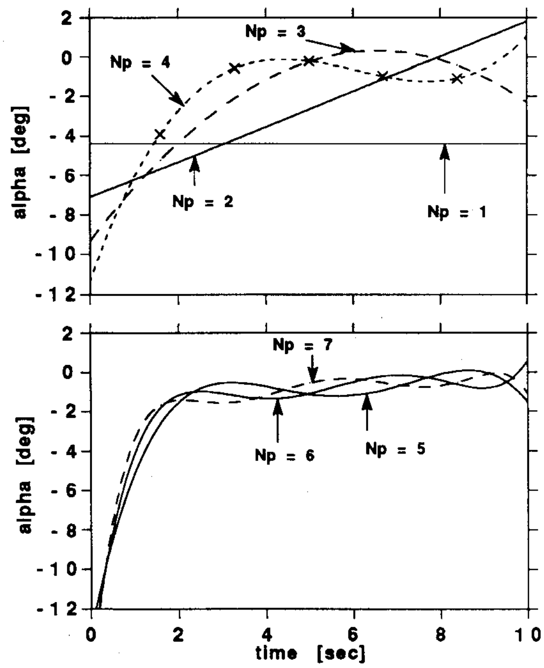

Fig. 2 Time histories of angle of attack.

Numerical Methods

Two numerical updating schemes follow naturally from the expressions of δJ and $\delta^2 J$. A steepest descent scheme from Eq. (7) is

$$\Delta\pi = -K J_\pi^T = -K \left[\phi_\pi + \int_{t_0}^{t_f} H_\pi \, \mathrm{d}t \right]^T \tag{16}$$

where $K > 0$ is a stepsize. If the second-order gradient matrix P is positive-definite, a second-order update scheme can be used

$$\Delta\pi = -P^{-1} J_\pi \tag{17}$$

To start these algorithms, one needs to guess an initial parameter π_0. At each iteration, one needs to integrate Eq. (2) forward to obtain $x(t)$, integrate Eq. (6) backward to obtain $\lambda(t)$, and compute J_π from Eq. (8). If the second-order method is used, one also needs to calculate $A(t, t_0)$ from Eq. (12) and P from Eq. (14). In both algorithms, the stopping criterion can be $\|J_\pi\| = \sqrt{J_\pi J_\pi^T} < \epsilon$, where $\epsilon > 0$ is a preselected small number.

Evaluations of the first- and second-order gradients using the previous expressions require many fewer integrations than those performed by digital differentiation. In terms of the number of equivalent numerical integrations of a scalar differential equation, the previous expressions need $(2+p)n + \frac{1}{2}p(p+3)$ at each iteration, whereas a forward difference scheme needs $\frac{1}{2}(n+1)(p+1)(p+2)$. At each iteration, the previous expressions offer a saving of

$$\tfrac{1}{2} n(p-1)(p+2) + 1 \tag{18}$$

With $n = 2$ and $p = 7$, for example, this number is 55! In addition, this saving increases drastically as the number of parameters and/or states increases. A central difference scheme needs even more integrations than a forward difference scheme.

The sequential gradient restoration algorithm[5,6] by Miele et al. may also be used to solve parameter optimal control problems. This algorithm has two phases and is different from the proposed method in this paper. In particular, one needs initially to guess both the unknown parameters and a nominal state history.

Example

Vincent and Grantham[1] and Stengel[2] both provide some analytical examples of parameter optimal control. The following numerical example demonstrates the use of parameter optimal control in trajectory optimization.

A trajectory optimization problem can be converted into a parameter optimal control format with different control parameterizations. There are three ways of parameterizing control functions. The most common one is to approximate a control function by piecewise constant (Goh and Teo[7]) or general spline functions. Control functions can also be approximated by polynomials or other functions of time. The validity of this parameterization is guaranteed by the Weierstrass theorem.[8] In addition, one can assume special parameterization forms for controls.

The equations of an ascending rocket are given[2] as

$$\dot{v} = \frac{T}{m}\cos\alpha - g\sin\gamma \tag{19}$$

$$v\dot{\gamma} = \frac{T}{m}\sin\alpha - g\cos\gamma \tag{20}$$

where v is the flying speed in ft/s, γ is the flight path angle defined with respect to the horizontal plane, and α is the angle between the thrust vector and the velocity vector. The control angle $\alpha(t)$ is approximated by a polynomial function of time

$$\alpha(t) = \alpha_0 + \alpha_1 t + \cdots = \sum_{k=0}^{N_p-1} \alpha_k t^k \tag{21}$$

The problem then is to determine parameters α to minimize

$$I = \left(\gamma_f - \frac{\pi}{3}\right)^2 + \int_0^{t_f} \alpha^2(t)\,\mathrm{d}t \tag{22}$$

The initial conditions are $v(0) = 100$ ft/s and $\gamma(0) = \pi/2$. Other parameters are $t_f = 10$ s, $T = 10{,}000$ lbs, $m = 20$ slugs, and $g = 32$ ft/s^2.

The second-order algorithm is used with the initial guesses of $\alpha_k = 0$, $k = 0, 1, \ldots, N_p$. Figure 1 shows the reductions in the optimal cost as the number of parameters increases. Figure 2 compares $\alpha(t)$ histories for cases of up to seven parameters. A further increase in the number of parameters does not decrease the cost very much.

Conclusions

This Note employs two basic approaches and presents the necessary and sufficient conditions for an unconstrained parameter optimal control problem. Two numerical solution methods are also given. An example problem is used to demonstrate the use of parameter optimal control and polynomial parameterization in trajectory optimization. Use of the gradient expressions in numerical solutions avoids the large number of integrations needed in digital differentiations.

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Book Announcements

KAUFMAN, H., BAR-KANA, I., and SOVEL, K., ***Direct Adaptive Control Algorithms,*** Springer-Verlag, New York, 1994, 374 pages, $69.00.

Purpose: This text contains an in-depth review of adaptive control techniques that have been developed by the authors over the last decade. The monograph presents a rigorous treatment of multi-input, multi-output control theory as well as a thorough discussion of algorithmic implementation features and applications.

Contents: Output model following; stability and positivity; nonlinear adaptive controllers; positive real analysis; parallel feedforward control; model reference adaptive control; robust redesign of adaptive algorithms; robustness considerations with feedforward in the reference model; design of MRAC systems; case studies.

ESTRADA, R., and KANWAL, R. P., ***Asymptotic Analysis: A Distributional Approach,*** Birkhauser, Cambridge, MA, 1994, 258 pages, $64.95.

Purpose: This reference provides a theoretical framework for asymptotic expansions encountered in the solution of a large class of problems in engineering and physics.

Contents: Asymptotic series; algebraic and analytic operations; space of distributions; regularizations; distributional derivatives; tempered distributions; expansion of oscillatory kernels; multidimensional generalized function; series of Dirac delta functions.

GRAY, W., LEIJNSE, A., KOLAR, R. L., and BLAIN, C. A., ***Mathematical Tool for Changing Spatial Scales in the Analysis of Physical Systems,*** CRC Press, Boca Raton, FL, 1993, 323 pages, $39.95.

Purpose: This book contains comprehensive introduction to fundamental mathematics for changing spatial scale in physical models. As such, it is a welcome addition to the growing literature applicable to homogenization as encountered in damage mechanics and mechanics of composite materials.

Contents: Scale and derivation of balance laws; generalized functions of curves; surfaces and volumes; integration scales; integration region selection with generalized functions; derivation of averaging theorems.

DE BOOR, C., HOLLIG, K., and REIMANSCHNEIDER, S., ***Box Splines,*** Springer-Verlag, New York, 1994, 200 pages, $34.00.

Purpose: This book describes the mathematical theory of multivariable splines that encompasses many results that have, until now, only been available in journal articles.

Contents: Definitions of box splines: analytic, geometric, inductive; recurrence relations; zonotopes; linear algebra of box splines; quasi-interpolation and approximation powers; cardinal interpolation; difference equations; cardinal splines; wavelets; discrete box splines; subdivision algorithms.